BRANCHING OF THE SOLUTION OF THE EQUATIONS OF A SPHERICAL SHELL UNDER CONDITIONS OF A SINGULAR PERTURBATION

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An adaptive variational method is presented here and is applied to the study of the stress–strain state of a thin elastic shell. The method makes use of the maximum principle for inclusions [1, 2] and is characterized by the fact that the coordinate system is calculated on a computer. Meanwhile, in conformity with the notion advanced by G. I. Marchuk, the class of functions for which the above principle is realized, is assigned with allowance for a priori known properties of the solutions of the boundary-value problem [3, 4]. A model is proposed for kinematic relations of the geometrically nonlinear theory of shells in a "quadratic" approximation.

1. Formulation of the Problem. We will examine the system of Marguerre–Vlasov equations

$$
\begin{gathered}
\mu\Delta^2 w = \theta\Delta\Phi + \Phi'' H w + w'' H\Phi - 2I\Phi I w + p,\\
\mu\Delta^2\Phi = (Iw)^2 - w'' H w - \theta\Delta w,\ (r, \varphi) \in \Omega,\ \Omega = [0, 1) \times [0, 2\pi],\\
rH(\ldots) = (\ldots)' + r^{-1}(\ldots)^{\cdot\cdot},\ rI(\ldots) = r^{-1}(\ldots)^{\cdot} - (\ldots)^{\cdot\prime},\\
r \in \partial\Omega,\ w = w' = 0,\ H\Phi = I\Phi = 0,\\
\mu = h/a\gamma,\ \gamma^2 = 12(1 - \nu^2),\ \theta = a/R,\ (\ldots)' = \partial/\partial r(\ldots),\ (\ldots)^{\cdot} = \partial/\partial\varphi(\ldots),
\end{gathered}
\tag{1.1}
$$

describing the stress–strain state, stability, and postbranching behavior of a shallow spherical cap with a hinged contour. Here, w is the normal displacement of the middle surface; Φ is an Airy function; μ is a natural small parameter with higher derivatives; θ is the half-angle of the shell; a is the bearing radius; R is the radius of curvature of the sphere; p is the external pressure; h is the thickness of the shell; ν is the Poisson's ratio; all of the quantities are dimensionless. The procedure for changing over to the dimensional quantities is described in [5].

The problem being examined permits the use of a dual method of analysis whereby, together with the Marguerre–Vlasov problem, we employ its below variational formulation [6] to approximate the solution of equations of shell theory

$$
\begin{gathered}
\langle \operatorname{grad} Z(\mathscr{F}), \omega\rangle = 0,\ \forall\omega \in \mathbf{U}(\Omega),\ \mathbf{U}(\Omega) = \mathring{\mathbf{H}}^2(\Omega) \times \mathring{\mathbf{H}}^2(\Omega),\\
Z(\mathscr{F}) = \frac{1}{2}\int\limits_{\Omega} \{\mu[(\Delta\mathscr{F}_1)^2 - (\Delta\mathscr{F}_2)^2] - 2\mathscr{F}_1\Delta\mathscr{F}_2 + 2p\mathscr{F}_2 - \mathscr{F}_1 \times\\
\times [\mathscr{F}_1'' H\mathscr{F}_2 + \mathscr{F}_2'' H\mathscr{F}_1 - 2I\mathscr{F}_1 I\mathscr{F}_2]\}\, d\Omega, \quad \mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2) \equiv (\Phi, w).
\end{gathered}
\tag{1.2}
$$

The formulations (1.1) and (1.2) are equivalent in a Hilbert space of two-dimensional vector-functions $\mathbf{U}(\Omega)$. When the solution constructed in an attempt to develop approximate methods of analysis contains an approximation error, the accuracy of the methods can be improved by minimization of the function Z(U).

In deriving equations of shell theory, we used static relations in the form of Hooke's law, the Kirchoff–Love hypothesis, and kinematic expressions

$$
\begin{gathered}
\varepsilon_\alpha = A^{-1}[u'_\alpha + B^{-1}A'_\beta v] - k_\alpha w + (1/2)\,\omega_\alpha^2,\quad A\omega_\alpha = -w'_\alpha,\\
\varepsilon_\beta = B^{-1}v'_\beta + B'_\beta u - k_\beta w + (1/2)\,\omega_\beta^2,\quad B\omega_\beta = -w'_\beta,\\
\varepsilon_{\alpha\beta} = BA^{-1}(B^{-1}v)'_\alpha + AB^{-1}(A^{-1}u)'_\beta + \omega_\alpha\omega_\beta,
\end{gathered}
\tag{1.3}
$$

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where $A(\alpha, \beta)$, $B(\alpha, \beta)$ are coefficients of the first quadratic form of the undeformed middle surface; α and β are curvilinear coordinates; u and v are the tangential displacements in the directions α and β; k_α and k_β are the principal curvatures of the surface in question. Thus, within the framework of the shell-theory model in strain tensor components being studied here, we retain nonlinear terms only for the angles of rotation of the normal to the shell surface when it is rotated in two planes: α = const, β = const.

In 1939-1942, Friedrichs and Stoker discovered a new effect — a circular plate subjected to uniform compression at high pressures has a region in which tensile forces are present. This phenomenon was unexpected for such plates, but the investigations were able to substantiate their findings by the asymptotic method [7, 8]. A similar result was obtained in [9] in formulating a precise experiment concerning the stability of a spherical segment under a uniform external pressure p.

The paradoxicality and importance of these phenomena are related to the fact that, as an example, the sphere in the case being studied here might become unstable in nonaxisymmetric modes. In the opinion of specialists, such bifurcation of the solution is due to the fact that appreciable compressive forces are developed in the region of large strains. This notion has been used as the basis for the current explanation of the existence of nonaxisymmetric loss of stability [10].

The authors of [9] examined shells only at $\mu \geq 1.07 \cdot 10^{-3}$, since it is very difficult to study the stress—strain state in a precision experiment at smaller values of μ. Here, we propose to analyze this problem by approximate methods, having expanded the range of variation of μ. However, several fundamental questions arise in approximating the solution [11-13] when these methods are employed.

The method to be used is based on the notion [3, 4] that the efficiency of projection methods can be increased if the solution in the high-gradient region is approximated by two sequences $\{\varphi_i\}$ and $\{\psi_i\}$, one of which takes into account the singular properties of the solution of the shell-theory equations.

2. Method of Solution and Results of Computation. As the first sequence of test spaces $\{\mathbf{V}_n\}$, $\mathbf{V}_1 \subset \mathbf{V}_2 \subset \mathbf{V}_3, \ldots, \mathbf{V}_{n-1} \subset \mathbf{V}_n$ we take Hilbert spaces satisfying the requirement of smoothness of the solution of the initial problem. However, generally speaking, the conditions for $\partial\Omega$ need not be satisfied for the elements of these spaces. Here, $n = \dim \mathbf{V}_n$. We specify the rule for selection of the basis functions $\{\varphi_i\}_1^n$ in $\mathbf{V}_n$ for each fixed n.

The first algorithm involves the construction of the n-th approximation $(\eta_n = (w_n, \Phi_n))$ in the form

$$\eta_n = \sum_{i=1}^{n} c_i \circ \varphi_i, \quad c_i = (c_i^1, c_i^2), \quad c \circ \varphi \stackrel{\mathrm{def}}{=} (c^1\varphi^1, c^2\varphi^2), \tag{2.1}$$

where the coefficients of the expansion c_i are determined from Bubnov's nonlinear algebraic system

$$f_i(c_1, c_2, c_3, \ldots, c_n, p) = 0, \ i = \overline{1, 2n}. \tag{2.2}$$

The explicit expressions for the function f_i are not written out because the procedures for obtaining them are well known.

Let ξ be a parameter $(-\infty < \xi < 0)$. We introduce the variable $\tau = \mu^\xi \times (1 - r)$. Proceeding as above, we specify another sequence of Hilbert spaces $\{\mathbf{H}_k\}$, $\mathbf{H}_1 \subset \mathbf{H}_2 \subset \mathbf{H}_3, \ldots, \mathbf{H}_{k-1} \subset \mathbf{H}_k$. First of all, these spaces belong to the prescribed class in the sense of the smoothness limitation. Secondly, they make it possible to compensate for the error in the boundary conditions and part of the bearing contour of the shell. We will designate the basis functions in $\mathbf{H}_k$ through $\{\psi_j\}_1^n$, where $\psi_j = \mathscr{P}^j\rho_j$, $\mathscr{P}^j$ is the polynomial form of τ, $\rho_j = \exp(-\lambda_j\tau)$, $\infty > \mathrm{Re}\lambda_j > \delta > 0$. It is assumed that the exponential multiplier ρ_j for each term of the sequence of test spaces $\{\mathbf{H}_k\}$ is explicitly determined, although the numerical value of the parameters λ_j may be unknown.

The second algorithm consists of the expansion $\mathbf{H}_{kn} = \eta_n + (w_{kn}, \Phi_{kn})$ in two systems of functions

$$\mathbf{H}_{kn} = \sum_{i=1}^{n} c_i \circ \varphi_i + \sum_{j=1}^{n} d_j \circ \psi_j, \quad d_j = (d_j^1, d_j^2) \tag{2.3}$$

(d_j are unknown coefficients). To determine them, we insert (2.3) into the shell-theory equations and boundary conditions after we express r in the coefficients and differential operators through τ. We group all of the expressions which do not contain the elements ψ_j or their derivatives. We then drop all of the above-indicated terms, considering that series (2.1) satisfies the equations of a sphere with the necessary accuracy ε on Ω — except perhaps for a small region in the neighborhood of the bearing contour — and that the expressions obtained are stable against ε-perturbations. After these simplifications, each of the remaining terms in the equation of the spherical shell will be exponentially small at large τ. Having projected the thus-transformed expressions onto elements of the sequence $\{\psi_j\}_\sigma^n$, we find Bubnov's second nonlinear algebraic system

$$g_i(c_1, c_2, c_3, \ldots, c_n, d_1, d_2, d_3, \ldots, d_k, \lambda_1, \lambda_2, \ldots, \lambda_k) = 0,\ i = \overline{1, 2(k-\sigma)}. \tag{2.4}$$

Such use of the second Bubnov process leads to an indeterminate algebraic system in the unknowns $d_1, d_2, \ldots, d_k$, since (2.4) contains 2k coefficients $d_j = (d_j^1, d_j^2)$ and $2(k - \sigma)$ equations. The additional 2σ equations are derived from the boundary conditions. We join them to the Bubnov system and henceforth consider (2.4) to represent the expanded algebraic problem.

If the indices of the exponents $\lambda_1, \lambda_2, \ldots, \lambda_k$ are known, then we construct an approximate solution $\mathbf{H}_{kn}$ by successively solving (2.2) and (2.4) with the use of (2.3). However, the question of the numerical value of $\lambda_1, \lambda_2, \ldots, \lambda_k$ remains open in nonlinear problems of shell theory, except for rare exceptions.

Third Algorithm. To close the proposed method, we insert (2.3) into the expression for the energy functional. Then from the condition

$$Z \to \inf_{\lambda_j} \tag{2.5}$$

we easily find the remaining k equations. In the minimization of the functional Z, these equations might be Ritz equations if there were no limitations on λ_j. The last fact necessarily involves use of the maximum principle. To use this principle in the traditional form, it suffices to reduce algebraic problems (2.2), (2.4) to Cauchy problems with governing parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$ from the region of permissible values, having differentiated (2.2) and (2.4) with respect to the pressure p. In this case, the following integral may be the quality functional

$$J = \int_0^p Z\, ds,$$

while the initial conditions may be trivial values for c_i, d_j at $p = 0$.

One of the shortcomings of the method has to do with the awkwardness of the derivation of the integrated equations. The preliminary analytical calculations necessary for realizing the projection method can be simplified if the resulting systems of differential equations are not solved for the derivatives $(c_i)'_p$ and $(d_j)'_p$ and if this procedure is instead accomplished numerically on a computer. In this case, the quantities in question will contain rounding errors. Then correct application of the maximum principle will depend on the corresponding theorem not for the differential equations but for the sequences [1, 2]. Also, it proves possible to use a computer in the preliminary calculations when constructing the scalar product, although here it is necessary for small λ_j to calculate terms exceeding the numbers allowed by the words used. Thus, in the programmed realization, the range for the governing parameters is narrowed, and they cannot take values less than about 0.1 on a BÉSM-6 computer.

To continue the solution with respect to the parameter $p \in [0, p^*]$, where p^* is the branch point, we specify the directed sequence $\{p^i\}$ on the segment $[0, p^*]$. The solution between any two terms is found by the Runge–Kutta method. It is refined at points p^i by Newton's method. The location of these points is determined on the basis of the rate of convergence of the Newtonian iterations.

In the numerical integration of the equations of thin, geometrically nonlinear shells, it is interesting to examine the advance of possible small values of μ into the region if use of the method of small parameters with higher derivatives cannot be substantiated. At present, it is usually at least necessary to modify the method of calculation [14] with a reduction in μ by a factor of 2-4.

Below we present results of numerical analysis of the stress–strain state at $\mu = 2\cdot10^{-19}$, which is close to the natural limit – the roundoff of the BESM-6 computer. Our investigations was limited to the axisymmetric formulation

$$r \in (0, 1),\ \mu\mathscr{A}f + \theta\omega r + (1/2)\omega^2 = 0, \qquad (2.6)$$
$$\mu\mathscr{A}\omega - \Theta/r - f\omega - (1/2)pr^2 = 0,$$

$$\mathscr{A}(\cdots) = r\frac{d}{dr}r^{-1}\frac{d}{dr}r(\cdots),\quad \Phi = \int_0^r f(t)\,dt,\quad w = \int_1^r \omega(t)\,dt\ \ (r = 0, 1,\ f = \omega = 0).$$

In the first Bubnov process, to approximate each of the unknowns f and ω we assigned one power basis $(1 + r^2) \times \{r^{2k+1}\}_0^n$. In the projection of system (2.6) in $L_2(0, 1)$, we used elements of the form $\{r^{2k+1}\}_0^n$. In the second Bubnov process, the analogous coordinate sequences were equal to $\{\tau^k\}_1^d \exp(-\lambda\tau)$ and $\{\lambda^k\}_1^d$, respectively, where $\tau = (1 - r)\mu^{-1/2}$.

The values of the first four coefficients c_k for the stress function are as follows: 0.132616; 0.120845; –0.079648; 0.024718 at k = 0, 1, 2, and 3. Here and below, $\theta = 0.15$, $p = p^*$, and n = 3. The characteristic features of expansion (2.3) are clear – a relatively slow decrease in the coefficients c_k and alternation of signs with a change in k. This suggests that the accuracy of the results of the first Bubnov process is not high.

Table 1 shows points r_k over the meridian of the shell and values of $f(r_k)$ calculated in the approximation of the first Bubnov process. Also shown are results for the formal asymptotic solution at $\mu \to 0$ in Eqs. (2.6).

It is evident that, with such a decrease in c_k, the degenerate problem is determined with an error no greater than 0.6% when k increases. The error is as low as 0.15% at the point r = 1. The high accuracy at the boundary point is very important to the efficiency of the method, since it involves matching the solutions of two Bubnov processes in which the error of the calculations of c_k becomes the error of d_k.

Table 2 shows d_k for the stress function and angle of rotation of the normal of the shell element. Five terms are kept in the last sum in expansion (2.3). It is evident that d_k rapidly decreases with an increase in k. Thus, series (2.3) "converges internally" for both Bubnov processes.

In the above calculations, the parameter λ, characterizing the optimum weight of the test space H_4, turned out to be equal to 0.41375. Its value was determined from the condition of the potential energy minimum for the approximation in question.

In the realization of the projection method in high approximations, problems (2.2) and (2.4) proved to be ill-conditioned in the sense that ill-conditioned linear algebraic equations appeared in Newton's method. This was established from analysis of singular numbers of the corresponding matrices. Here, the rounding errors were taken into account by the method proposed in [15], while the program pack necessary to do this was made available by the author of [16].

Figures 1 and 2 show the forces in the middle surface of the shell. The following laws can be established: 1) N_φ and N_r are nonmonotonic functions of the variable τ; 2) there is a point A at which a uniaxial stress state exists; 3) the boundary-layer region contains a circle $T = \{\tau < 6.4 | N_\varphi > 0\}$ such that the sphere is tensioned in it; 4) although the point of localization of the maximum forces B is located near the boundary $\partial\Omega$, it is not part of the boundary.

On the whole, the relation $N_\varphi \gg N_r$ is valid for the stress state in the edge-effect region. Thus, the force N_φ determines the strain of the spherical shell in this region at small μ.

TABLE 1

r_k	$-10f(r_k)$	$10f(r_k),\ \mu\to 0$	r_k	$-10f(r_k)$	$-10f(r_k),\ \mu\to 0$
0,1	0,132726	0,133566	0,6	0,802970	0,801576
0,2	0,266048	0,267192	0,7	0,934706	0,935172
0,3	0,400139	0,400788	0,8	1,068087	1,068764
0,4	0,534612	0,534384	0,9	1,203048	1,202364
0,5	0,668737	0,667983	1,0	1,334005	1,335960

TABLE 2

k	d_k for f	d_k for ω	k	d_k for f	d_k for ω
0	0,1334005	0,0000000	3	0,0106243	0,0080705
1	0,1670084	0,0249054	4	−0,0002632	−0,0002400
2	0,1040459	−0,0579592			

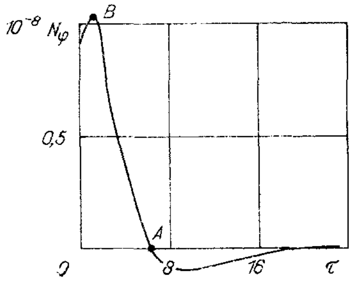

Fig. 1

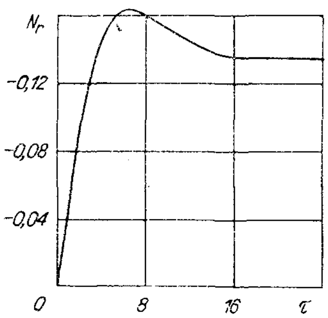

Fig. 2

The above-described method was used to find the critical pressure $p^* = 0.4454$, which is only 3.8% greater than the results obtained by the range method for μ as small as possible. Repeated attempts were made earlier to calculate it with $\mu \in [\mu_0, 0)$, $\mu_0 \ll 1$. However, these attempts were not successful, since there was no method of approximating the boundary-layer equations in the neighborhood of branch points and it was necessary to solve ill-conditioned systems of algebraic equations. The method developed here makes it possible to obtain an approximation, uniform with respect to μ of the solution of the Marguerre–Vlasov equations for a sphere. Realization of the method on a computer showed that it also answers the question of the numerical integration of these equations in the region of small μ in the axisymmetric formulation.

We also determined the sequence of points of nonaxisymmetric branching $\{p_\ell\}$ and the critical pressure $p_c = \min_\ell \{p_\ell\}$. Here, p_ℓ is the eigenvalue of the boundary-value problem

$$
\begin{gathered}
\mu\Delta_l^2 W_l = \theta\Delta_l\Phi_l + f'H_lW_l + \beta' H_l\Phi_l + \left(\beta\Phi_l'' + fW_l''\right) r^{-1},\\
\mu\Delta_l^2\Phi_l = -\theta\Delta_l W_l - \beta' H_l W_l - \beta W_l'' r^{-1}, \quad r = 1, \quad W_l = W_l' = 0,\\
H_l\Phi_l = I_l\Phi_l = 0,\\
rH_l(\ldots) = (\ldots)' - l^2 r^{-1}(\ldots),\ rI_l(\ldots) = -l[(\ldots)r^{-1} - (\ldots)'],\\
\Delta_l(\ldots) = (\ldots)'' + (\ldots)'r^{-1} - l^2 r^{-2}(\ldots),
\end{gathered}
$$

which is obtained from (1.1) by expansion of the solution into a Fourier series and its linearization in the neighborhood of the axisymmetric solution (β, f). The indices of the exponents λ_k^ℓ of the basic functions are found from the condition

$$
G_n^l \to \inf_{\{\lambda_k^l\}}, \quad G_n^l \stackrel{\text{def}}{=} G(W_l^n, \Phi_l^n),
$$

TABLE 3

μ	p_c	l	μ	p_c	l	μ	p_c	l
$\theta=0,6$			$\theta=0,8$			$\theta=1,0$		
0,013			0,013	0,7356	2	0,013	0,7185	3
0,011	0,7668	2	0,011	0,7290	3	0,011	0,7034	4
0,009	0,7428	2	0,009	0,7305	3	0,009	0,7190	4
0,007	0,7470	2	0,007	0,7210	4	0,007	0,7519	4

$$G(W_l, \Phi_l) = \frac{1}{\pi}\int_0^1 \rho(r)\left[\mu\left(W_l\Delta_l^2 W_l - \Phi_l\Delta_l^2\Phi_l\right) - W_l\left(f' H_l W_l + f W_l'' r^{-1}\right) - \right.$$

$$\left. - 2W_l\left(\theta\Delta_l\Phi_l + \beta' H_l\Phi_l + r^{-1}\beta\Phi_l''\right)\right] r\,dr,$$

$$W_l^n = \sum_{k=2}^{n} d_{k,l}^1\psi_k^l, \quad \Phi_l^n = \sum_{k=2}^{n} d_{k,l}^2\psi_k^l, \quad \psi_k^l = \tau^k\exp\left(-\lambda_k^l\tau\right),$$

where $\rho(r)$ is the weight function; $d_{k,\ell}^i$ are known coefficients of the expansion ($i = 1, 2$).

Calculations were performed for $\mu = 8.5\cdot 10^{-5}$, $\theta = 0.15$, $n = 6$, $\rho(r) = r^4$, $\lambda_2^\ell = \lambda_3^\ell = \ldots \lambda_6^\ell = \lambda^\ell$. It turned out that the value of λ^ℓ at which the functional G_6^ℓ reaches its minimum is 0.300, while the corresponding critical pressure and wave-formation parameter takes values of 0.2339 and 20, respectively.

3. Kinematic Relations. Derivation of the equations of the geometrically nonlinear theory of shells generally requires the satisfaction of two restrictions: the equations should be simple enough to permit numerical integration; all of the nonlinear terms determining the stress–strain state must be kept in the strain tensor.

In conformity with the method being examined here, the following plan is used in formulating the boundary-value problem. Kinematic relations are assigned on the basis of mechanical considerations. Then, in accord with [17], we derive the equilibrium and strain-compatibility equations.

Table 3 shows critical values of the pressure p_c on a spherical shell with a fixed edge. Also shown are the corresponding numbers of the harmonics in the Fourier series ℓ. The geometry of the middle surface was not identified with a plane [18].

It is evident that in the investigated range of μ and θ, the number of harmonics ℓ does not exceed four. For $\ell \leq 3$, some of the expressions omitted from the fuller representation of the strain tensor are comparable to the quadratic terms in (1.3). If we replace these expressions, then two new terms appear in the kinetic relations

$$\begin{aligned} \varepsilon_\alpha &= A^{-1}\left[u_\alpha' + B^{-1}A_\beta' v\right] - k_\alpha w + k_\alpha^2 w^2 + (1/2)\,\omega_\alpha^2, \\ \varepsilon_\beta &= B^{-1}v_\beta' + B_\beta' u - k_\beta w + k_\beta^2 w^2 + (1/2)\,\omega_\beta^2, \\ AB\varepsilon_{\alpha\beta} &= B^2\left(B^{-1}v\right)_\alpha' + A^2\left(A^{-1}u\right)_\beta' + \omega_\alpha\omega_\beta. \end{aligned} \tag{3.1}$$

At $\mu < 0.007$, it can be suggested that the strains from (3.1) and (1.3) are close because the shell is deformed mainly in a certain region of the edge under conditions of a singular perturbation. At the same time, the angles of rotation of the normal of a shell element in this case satisfy the inequalities $|\omega_\eta| \gg k_\eta w$, $\eta = \alpha, \beta$. However, numerical analysis shows that within the framework of the given model, results obtained previously are also refined in this region if we consider the entire sequence $\{p_\ell\}$, since branching of the solution at $p_\ell \gg p_c$ is possible for small ℓ.

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